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LOCAL-IN-SPACE BLOWUP CRITERIA FOR A CLASS OF NONLINEAR DISPERSIVE WAVE EQUATIONS

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1. Introduction

In this note we will discuss blowup issues for solutions of the non-linear dispersive wave equation on the real line

$$u_t - u_{txx} + [f(u)]_x - [f(u)]_{xxx} + \Big[g(u) + \frac{f''(u)}{2}u_x^2\Big]_x = 0, \qquad x \in \mathbb{R},\ t > 0.. \tag{1.1}$$

Equation (1.1) is often referred as the *generalized hyper-elastic rod wave equation*, see [19]. It includes several models of considerable physical interest:

- The propagation of non-linear waves inside cylindrical hyper-elastic rods, assuming that the diameter is small when compared to the axial length scale, is described by the one dimensional equation (see [15]),

  $$v_\tau + \sigma_1 v v_\xi + \sigma_2 v_{\xi\xi\tau} + \sigma_3(2v_\xi v_{\xi\xi} + v v_{\xi\xi\xi}) = 0, \qquad \xi \in \mathbb{R},\ \tau > 0.$$

  Here $v(\tau,\xi)$ represents the radial stretch relative to a pre-stressed state, $\sigma_1 \neq 0$, $\sigma_2 < 0$ and $\sigma_3 \le 0$ are physical constants depending on the material. The scaling transformations

  $$\tau = \frac{3\sqrt{-\sigma_2}}{\sigma_1}t, \qquad \xi = \sqrt{-\sigma_2}x,$$

  with $\gamma = 3\sigma_3/(\sigma_1\sigma_2)$ and $u(t,x) = v(\tau,\xi)$, allow us to reduce the above equation to

$$u_t - u_{xxt} + 3uu_x = \gamma(2u_x u_{xx} + u u_{xxx}), \qquad x \in \mathbb{R},\ t > 0. \tag{1.2}$$

  The rod equation (1.2) thus corresponds to the choice $f(u) = \frac{\gamma}{2}u^2$ and $g(u) = \frac{3-\gamma}{2}u^2$.
- If we choose instead $f = 0$ and $g = u + \frac{1}{2}u^2$ we get the BBM equation

  $$u_t + u_x + uu_x - u_{txx} = 0,$$

  proposed by Benjamin, Bona and Mahony as an improvement of the celebrated KdV equation for the modelling long surface gravity waves of small amplitude, with better stability properties at high wavenumbers.

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- The choice $f(u) = u^2/2$, $g(u) = \kappa u + u^2$ leads to the Camassa–Holm equation [8,9]:

$$u_t + \kappa u_x - u_{xxt} + 3uu_x = uu_{xxx} + 2u_x u_{xx}, \qquad x \in \mathbb{R}, \quad t > 0. \tag{1.3}$$

As it is the case for the KdV equation, the Camassa–Holm equation can be derived as an asymptotic model for water wave propagation in channels from the free surface Euler equations, in the so called shallow water regime $\mu = h^2/\lambda^2 \ll 1$, where h and λ denote respectively the average elevation of the liquid over the bottom and the characteristic wavelength. Equation (1.3) models small, but finite, amplitude waves, *i.e.* wave such that the dimensionless amplitude parameter $\epsilon = a/h$ satisfies $\epsilon = O(\sqrt{\mu})$, where a is the typical amplitude. Notice that the rigorous derivation of the KdV would require the more stringent scaling $\epsilon = O(\mu)$. Thus, the Camassa–Holm equation is more accurate to study the propagation of larger amplitude waves. Both non-linear and dispersive effects are present, but for larger amplitude waves non-linear effects become preponderant and wave breaking can eventually occur. Not surprisingly, the KdV equation is not suitable for the description of breaking mechanisms, as its solutions remain smooth for all time. On the other hand, many blowup results are available for the Camassa–Holm equation: see, *e.g.* [3, 9, 11, 12, 23, 26]. Such equation attracted a considerable interest in the past 20 years, not only due its hydrodynamical relevance (it was the first equation capturing both soliton-type solitary waves as well as breaking waves) but also because of its extremely rich mathematical structure.
- When $f(u) = \frac{u^{Q+1}}{Q+1}$ and $g(u) = \kappa u + \frac{Q^2+3Q}{2(Q+1)} u^{Q+1}$ one recovers from (1.1) another class of equations with interesting mathematical properties, studied in [18].

From now on, we will study the Cauchy problem for the generalized rod equation, written in the non-local form, formally equivalent to (1.1):

$$\begin{cases} u_t + f'(u)u_x + \partial_x p * \left[g(u) + \frac{f''(u)}{2}u_x^2\right] = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1.4}$$

Here

$$p(x) = \tfrac{1}{2}e^{-|x|}$$

is the fundamental solution of the operator $1 - \partial_x^2$. The problem (1.4) is thus written in the abstract form

$$\frac{du}{dt} + A(u) = H(u), \qquad u(x,0) = u_0(x),$$

with $A(u) = f'(u)\partial_x$ and $H(u) = -\partial_x(1-\partial_x^2)^{-1}\left[g(u) + \frac{f''(u)}{2}u_x^2\right]$. The local existence theory can be developped applying classical Kato's approach [21]. For reader's convenience we collect in a single theorem the main results of the recent paper of Tian, Yan and Zhang [24] on the problem (1.4).

Theorem 1.1 (See [24]).

(1) *Assume that $f, g \in C^\infty(\mathbb{R})$. Let $u_0 \in H^s(\mathbb{R})$, $s > 3/2$. Then there exists $T > 0$, with $T = T(u_0, f, g)$ and a unique solution u to the Cauchy problem* (1.4) *such*

that $u \in C([0,T), H^s(\mathbb{R})) \cap C^1([0,T), H^{s-1}(\mathbb{R}))$. *The solution has constant energy integral*

$$\int_{\mathbb{R}} (u^2 + (u_x)^2) = \int_{\mathbb{R}} (u_0^2 + (u_0')^2) = \|u_0\|_{H^1}^2.$$

Moreover, the solution depends continuously on the initial data: the mapping $u_0 \mapsto u$ *is continuous from* $H^s(\mathbb{R})$ *to* $C([0,T^*), H^s(\mathbb{R})) \cap C^1([0,T), H^{s-1}(\mathbb{R}))$.

(2) *Assume in addition that* $f'' \geq \gamma > 0$:

i) *(Blowup scenario and rate) Let* $0 < T^* \leq \infty$ *be the maximal time of the solution in* $C([0,T), H^s(\mathbb{R})) \cap C^1([0,T^*), H^{s-1}(\mathbb{R}))$. *Then* $T^* < \infty$ *if and only if*

$$\lim_{t \to T^*} \inf_{x \in \mathbb{R}} u_x(t,x) = -\infty.$$

In this case, the blowup rate of $\inf_{x\in\mathbb{R}} u_x(t,x)$ *as* $t \to T^*$ *is* $O(\frac{1}{T^*-t})$.

ii) *(Blowup criterion) Assume that there exists a point* $x_0 \in \mathbb{R}$ *such that*

$$u_0'(x_0) < -\sqrt{\frac{2C_0}{\gamma}}, \tag{1.5}$$

where $C_0 = C_0(\|u_0\|_{H^1}, f, g)$ *is given by*

$$C_0 \equiv 2 \sup_{|v| \leq \|u_0\|_{H^1}} |g(v)| + \frac{\|u_0\|_{H^1}^2}{2} \sup_{|v| \leq \|u_0\|_{H^1}} f''(v).$$

Then u *blows up in finite time and* $T^* \leq \frac{1}{\sqrt{2C_0\gamma}} \log\left(\frac{\sqrt{\gamma/2}\,u_0'(x_0) - \sqrt{C_0}}{\sqrt{\gamma/2}\,u_0'(x_0) + \sqrt{C_0}}\right)$.

In the first item, the existence time T can be taken independent on the parameter s in the following sense: if u_0 also belongs to $H^{s_1}(\mathbb{R})$ with $s_1 > 3/2$, then we have also $u \in C([0,T), H^{s_1}(\mathbb{R})) \cap C^1([0,T), H^{s_1-1}(\mathbb{R}))$. Additional results in [24] include lower bound estimates for the existence time T^* and the lower semi-continuity of the existence time. Let us also mention the construction of global conservative weak solutions for such equation [19] (see also [6,7,10] for earlier results on weak solutions for more specific choices of the functions f and g).

2. Main results

The purpose of this note is to announce a new blowup criterion for equation (1.4) extending our previous result established in [3] in the special case of the classical rod equation. An expanded version of the present note, with complete proofs, will be published in [5]. Our second goal is to handle more general boundary conditions in order to encompass the case of solutions not necessarily vanishing at infinity. In particular, in the present note we will cover the case $f(u) = u^2/2$ and $g(u) = \kappa u + u^2$ corresponding to the Camassa–Holm equation with dispersion ($\kappa > 0$), a case that was not treated [3]. Contrary to previously known blowup criteria, like those in [9, 11, 12, 14, 22, 24–26], our criterion has the specific feature of being *purely local* in the space variable: indeed our blowup condition only involves the values of $u_0(x_0)$ and $u_0'(x_0)$ in a single point x_0 of the real line. On the other hand, for earlier criteria, checking the blowup conditions involved the computation of global quantities (typically, the $\|u_0\|_{H^1}$ norm, as in criterion (1.5) above) or other global conditions like antisymmetry assumptions or sign conditions on the associate potential.

As we shall see, in order to establish such blowup result we will need to restrict the choice of the admissible functions f and g. On the other hand, when available, our criterion is applicable to a wider class of initial data. We are now in the position of stating our theorem. Roughly speaking, under appropriate conditions on f and g, we get the finite time blowup as soon as

$$\exists\, x_0 \in \mathbb{R} \quad \text{such that} \quad u_0'(x_0) < -\beta\big|u_0(x_0) - c\big|, \tag{2.1}$$

where β and c are two real constants depending on the shape of the functions f and g.

We obtain two slightly different versions when g is bounded from below, or when g is bounded from above.

Theorem 2.1. *Let $f, g \in C^\infty(\mathbb{R})$ with $f'' \geq \gamma > 0$. Let $u_0 \in H^s(\mathbb{R})$, $s > 3/2$ Assume that at least one of the two following conditions* (1) *or* (2) *is fulfilled:*

(1) - *$\exists\, c \in \mathbb{R}$ such that $m = g(c) = \min_{\mathbb{R}} g$.*
- *The map $\phi\colon \mathbb{R} \to \mathbb{R}$ given by $\phi = \sqrt{\frac{1}{\gamma}(g - m)}$ is K-Lipschitz with $0 \leq K \leq 1$,*
- *$\exists\, x_0 \in \mathbb{R}$ such that $\quad u_0'(x_0) < -\frac{1}{2}\left(\sqrt{1 + 8K^2} - 1\right)|u_0(x_0) - c|$.*

(2) *Or, otherwise,*
- *$\exists\, c \in \mathbb{R}$ such that $M = g(c) = \max_{\mathbb{R}} g$.*
- *The map $\psi\colon \mathbb{R} \to \mathbb{R}$ given by $\psi = \sqrt{\frac{1}{\gamma}(M - g)}$ is K-Lipschitz with $0 \leq K \leq \frac{1}{\sqrt{8}}$*
- *$\exists\, x_0 \in \mathbb{R}$ such that $\quad u_0'(x_0) < -\frac{1}{2}\left(1 - \sqrt{1 - 8K^2}\right)|u_0(x_0) - c|$.*

Then the maximal time T^* of the solution u in $C([0, T^*), H^s(\mathbb{R})) \cap C^1([0, T^*), H^{s-1}(\mathbb{R}))$ of equation (1.4)*, constructed in Theorem 1.1, must be finite. An upper bound estimate for T^* is:*

$$T^* \leq \frac{4}{\gamma\sqrt{4u_0'(x_0)^2 - \left(\sqrt{1 \pm 8K^2} - 1\right)^2 \left(u_0(x_0) - c\right)^2}}, \tag{2.2}$$

where in the term $\pm 8K^2$ one has to take the positive sign under the conditions of Part (1) *and the negative sign under the conditions of Part* (2).

Here is a variant of the previous blowup result. First of all, notice that $H^s(\mathbb{R}) \subset W^{1,\infty}(\mathbb{R})$ for $s > 3/2$. Next Theorem applies more in general to solutions $u(t, \cdot) \in W^{1,\infty}(\mathbb{R})$. Therefore, it encompasses the case of infinite energy solutions, possibly non-vanishing as $x \to \infty$. We point out that a functional setting for constructing solutions with possibly different asymptotics as $x \to +\infty$ and $x \to -\infty$ was proposed in [17], at least in the case of the Camassa–Holm equation. But rather than choosing a functional setting where the solution can be constructed, we consider an *a priori* given solution:

Theorem 2.2. *Let $f, g \in C^\infty(\mathbb{R})$ with $f'' \geq \gamma > 0$. Let $u \in C([0, T^*), W^{1,\infty}(\mathbb{R})) \cap C^1([0, T^*), C^2(\mathbb{R}))$, with $0 < T^* \leq \infty$, be a solution of the generalized equation* (1.4)*. Assume that at least one of the conditions* (1) *or* (2) *of the previous theorem is fulfilled. Then T^* must be finite and is bounded from above by* (2.2).

In particular, if g is constant and u_0 is non-constant, then u blows up in finite time.

The technical C^2-regularity assumption is due to the lack of a well-posedness theory in the $W^{1,\infty}$ space. The complete proof of these two theorems will appear in [5]. Similar results hold true when the strict convexity assumption $f'' \geq \gamma > 0$ is replaced by the strict concavity condition $f'' \leq \gamma < 0$. In the latter case, the blowup criterion would be of the form $u_0'(x_0) > \beta\big|u_0(x_0) - c\big|$. Notice that, as expected, the case BBM equation, corresponding to the case $f \equiv 0$, is not covered by our theorems. Indeed, the solution of such equation are known to exist globally in time.

Applying this theorem we get as a corollary some information on the behavior at infinity of global solutions.

Corollary 2.3. *Let $f, g \in C^\infty(\mathbb{R})$ with $f'' \geq \gamma > 0$ be such that at least one of the two following conditions is satisfied (the maps ϕ and ψ are as in Theorem 2.1):*

(1) $\min_{\mathbb{R}} g = g(c)$ *and ϕ is K-Lipschitz with $0 \leq K \leq 1$, or otherwise*

(2) $\max_{\mathbb{R}} g = g(c)$ *and ψ is K-Lipschitz with $0 \leq K \leq \frac{1}{\sqrt{8}}$.*

(A) *If $u \in C([0, T^*), W^{1,\infty}(\mathbb{R}))$ is a smooth solution of the generalized rod equation* (1.1) *arising from an initial datum $u_0 \not\equiv c$ such that*

$$\liminf_{x \to +\infty} e^{\beta x}\big(u_0(x) - c\big) \leq 0 \quad \text{and} \quad \limsup_{x \to -\infty} e^{-\beta x}\big(u_0(x) - c\big) \geq 0, \tag{2.3}$$

where $\beta = \frac{1}{2}\big(\sqrt{1 + 8K^2} - 1\big)$ (in the case (1)), or $\beta = \frac{1}{2}\big(1 - \sqrt{1 - 8K^2}\big)$ (in the case (2)), then u must blow up in finite time.

In particular, if $u_0 \not\equiv c$ is such that $u_0(x) - c = o\big(e^{-\beta|x|}\big)$ for $|x| \to \infty$, then the corresponding solution of equation (1.1) *must blow up in finite time.*

(B) *If u is a global smooth solution in $C([0, \infty), W^{1,\infty}(\mathbb{R}))$ then for all $t \geq 0$,*

i) *Either $u(t, x) > c$ for all $x \in \mathbb{R}$,*

ii) *or $u(t, x) < c$ for all $x \in \mathbb{R}$,*

iii) *or $\exists\, x_t \in \mathbb{R}$ such that $u(t, \cdot) \leq c$ in $(-\infty, x_t]$ and $u(t, x) \geq c$ in $[x_t, +\infty)$. In this case, if $x \mapsto u(t, x)$ is equal to c at two distinct points of the real line, then $x \mapsto u(t, x)$ must be constant (equal to c) in the whole interval between them.*

We refer to [5] for the proof of this corollary. An obvious consequence in the case of periodic solutions is:

Corollary 2.4. *Under the conditions of the previous corollary for f and g, the identically constant solution $u \equiv c$ (where $c = \arg\min g$, or $c = \arg\max g$) is the only global, smooth and spatially periodic solution of the generalized rod equation* (1.1), *with time-independent energy integral on the torus, such that $u(t_0, x_0) = c$ for some $t_0 \geq 0$ and some $x_0 \in \mathbb{R}$.*

3. Outline of the proof of Theorem 2.1

We can assume, without restriction, that $s \geq 3$. Indeed, if $3/2 < s < 3$, and $u_0 \in H^s(\mathbb{R})$ satisfies a condition of the form (2.1), then we can approximate u_0 with a sequence of data belonging to $H^3(\mathbb{R})$ and satisfying the same condition (2.1), and next use the well-posedness result recalled in Theorem 1.1. In what follows we then consider a solution $u \in C([0, T^*) \cap H^3) \cap C^1([0, T^*), H^2)$. The blowup will result from the next elementary lemma:

Lemma 3.1 (See [4]). *Let $0 < T^* \leq \infty$ and $A, B \in C^1([0,T^*),\mathbb{R})$ be such that, for some constant $c_0 > 0$ and all $t \in [0,T^*)$,*

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}t}(t) &\geq c_0\, A(t)B(t) \\ \frac{\mathrm{d}B}{\mathrm{d}t}(t) &\geq c_0\, A(t)B(t). \end{aligned} \tag{3.1}$$

If $A(0) > 0$ and $B(0) > 0$, then

$$T^* \leq \frac{1}{c_0\sqrt{A(0)B(0)}} < \infty.$$

Next Lemma contains useful convolution estimates.

Lemma 3.2 (See [5]). *Let $\mathbf{1}_{\mathbb{R}^\pm}$ denote one of the two indicator functions $\mathbf{1}_{\mathbb{R}^+}$ or $\mathbf{1}_{\mathbb{R}^-}$.*

(1) *If f and g satisfy the condition as in Theorem 2.1 - (1) then the following estimate holds:*

$$(p\mathbf{1}_{\mathbb{R}^\pm}) * \Big(g(u) + \frac{f''(u)}{2}u_x^2\Big) \geq \frac{\alpha}{2}\Big(g(u) - m\Big) + \frac{m}{2} \tag{3.2}$$

with

$$\alpha = \frac{1}{4K^2}\Big(\sqrt{1+8K^2} - 1\Big). \tag{3.3}$$

(2) *If f and g satisfy the condition as in Theorem 2.1 - (2), then we have:*

$$(p\mathbf{1}_{\mathbb{R}^\pm}) * \Big(g(u) + \frac{f''(u)}{2}u_x^2\Big) \geq \frac{\alpha}{2}\Big(g(u) - M\Big) + \frac{M}{2} \tag{3.4}$$

with

$$\alpha = \frac{1}{4K^2}\Big(1 - \sqrt{1-8K^2}\Big). \tag{3.5}$$

In the case $g = m = M$ be a constant function (this corresponds to $K = 0$), the right-hand side of the above convolution estimates read $(p\mathbf{1}_{\mathbb{R}^\pm}) * (g + \frac{f''(u)}{2}u_x^2) \geq g/2$.

Consider the flow map, defined by

$$\begin{cases} q_t(t,x) = f'(u(t,q(t,x))), & t > 0, \quad x \in \mathbb{R}, \\ q(0,x) = x & x \in \mathbb{R}, \end{cases} \tag{3.6}$$

where u is the solution of the problem (1.4) given by Theorem 1.1. Notice that the assumptions made on f and u imply that $q \in C^1([0,T^*) \times \mathbb{R}, \mathbb{R})$ is well defined on the whole time interval $[0,T^*)$.

We now proceed putting the conditions (1) of our Theorems. Let us introduce

$$A(t) = \big(\beta(u-c) - u_x\big)(t, q(t,x_0)) \tag{3.7}$$

and

$$B(t) = \big(-\beta(u-c) - u_x\big)(t, q(t,x_0)). \tag{3.8}$$

Taking the space derivative in equation (1.4), and recalling that $(1-\partial_x^2)p$ equals the Dirac mass at the origin, we get

$$u_{tx} + f'(u)u_{xx} = -\frac{f''(u)}{2}u_x^2 + g(u) - p * \left[g(u) + \tfrac{f''(u)}{2}u_x^2\right]. \tag{3.9}$$

The kernel p satisfies the identity (both in the distributional and a.e. point-wise sense)

$$\partial_x p = p\mathbf{1}_{\mathbb{R}^-} - p\mathbf{1}_{\mathbb{R}^+}.$$

Let us set

$$\beta := \frac{1-\alpha}{\alpha} = 2K^2\alpha = \tfrac{1}{2}\left(\sqrt{1+8K^2}-1\right). \tag{3.10}$$

Then we get, recalling the inequality $f'' \geq \gamma$,

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t}(t) &= \beta(u_t + f'(u)u_x) - (u_{tx} + f'(u)u_{xx})\\ &= \frac{f''(u)}{2}u_x^2 - g(u) + (p - \beta\partial_x p) * \left(g(u) + \frac{f''(u)}{2}u_x^2\right)\\ &\geq \frac{\gamma}{2}u_x^2 - g(u) + (1+\beta)p\mathbf{1}_{\mathbb{R}^+} * \left(g(u) + \frac{f''(u)}{2}\right) + (1-\beta)p\mathbf{1}_{\mathbb{R}^-} * \left(g(u) + \frac{f''(u)}{2}\right).\end{aligned}$$

We now would like to apply the convolution estimates (3.2). This can be done, provided we have $-1 \leq \beta \leq 1$. Such additional condition is equivalent to $\alpha \geq 1/2$ and this last condition is ensured by the restriction $0 \leq K \leq 1$ made in the assumptions of Theorem 2.1-(1).

$$\begin{aligned}\frac{\mathrm{d}A}{\mathrm{d}t}(t) &\geq \frac{\gamma}{2}u_x^2 + (\alpha-1)\big(g(u)-m\big)\\ &= \frac{\gamma}{2}u_x^2 - (1-\alpha)\gamma\phi(u)^2\\ &\geq \frac{\gamma}{2}\left(u_x^2 - \beta^2(u-c)^2\right)\\ &= \frac{\gamma}{2}(AB)(t).\end{aligned} \tag{3.11}$$

Similar computations yield the estimate

$$\frac{\mathrm{d}B}{\mathrm{d}t}(t) \geq \frac{\gamma}{2}(AB)(t). \tag{3.12}$$

By our assumption on the initial datum made in Part (1) of Theorem 2.1,

$$u_0'(x_0) < -\frac{1}{2}\left(\sqrt{1+8K^2}-1\right)|u_0(x_0)-c|.$$

According to the definition of β (3.10), this can be re-expressed as

$$u_0'(x_0) < -\beta\big|u_0(x_0)-c\big|,$$

or, equivalently, as

$$A(0) > 0 \qquad \text{and} \qquad B(0) > 0.$$

Lemma 3.1 applies yielding the finite time blowup under conditions (1) of Theorem 2.1. The proof of Theorem 2.1 under conditions (2) and the proof of Theorem 2.2 require some slight changes, see [5] for details.

4. Application to the Camassa–Holm equation and the classical rod equations

The case $f(u) = \frac{1}{2}u^2$ and $g(u) = \kappa u + u^2$ corresponds to the Camassa–Holm equation with dispersion (1.3). Situation (1) of Theorem 2.1 applies (with $c = -\kappa/2$, $\phi(u) = \sqrt{u^2 + \kappa u + \kappa^2/4}$ and $K = 1$). We then immediately get the following corollary:

Corollary 4.1.

- *Let $u_0 \in H^s(\mathbb{R})$, with $s > 3/2$ be such that at some point $x_0 \in \mathbb{R}$ we have*
$$u_0'(x_0) < -\left|u_0(x_0) + \tfrac{\kappa}{2}\right|.$$
Then the corresponding solution of the Camassa–Holm equation breaks down in finite time.
- *If $u_0 \in C^2(\mathbb{R})$ is such that $u_0(x) + \kappa/2 = o(e^{-|x|})$ as $|x| \to \infty$, then the corresponding solution of the Camassa–Holm equations blows up in finite time.*
- *The identically constant solution $u \equiv -\kappa/2$ is the only global spatially periodic solution $u \in C([0,\infty), W^{1,\infty}(\mathbb{R}))$ of the Camassa–Holm equation (with time-independent energy integral on the torus) such that, for some $t_0 \geq 0$, and $x_0 \in \mathbb{R}$, $u(t_0, x_0) = -\kappa/2$.*

The result of the first item of the Corollary could be also obtained from the special case $\kappa = 0$ (established in [3]) via the change of unknown $v(t,x) = u(t, x - \frac{\kappa}{2}t) + \frac{\kappa}{2}$. Indeed, if u solves equation (1.3), then v solves the Camassa–Holm equation without dispersion. However, one should pay attention to the fact that the proof of [3] does indeed go through when applied to v, which requires slight changes (the point is that the solution v does not vanish as $x \to \infty$ as it was required in [3]).

In the case $f(u) = \frac{\gamma}{2}u^2$ and $g(u) = \frac{3-\gamma}{2}u^2$, corresponding to the classical rod equation (1.2), the conditions of our theorem are satisfied if and only if $1 \leq \gamma \leq 4$. Namely, situation (1) applies for $1 \leq \gamma \leq 3$ and situation (2) applies for $3 \leq \gamma \leq 4$. Making explicit our blowup condition in this case, we see that solutions of the classical rod equation break down in finite time as soon as, at some point $x_0 \in \mathbb{R}$, we have

$$u_0'(x_0) < -\frac{1}{2\sqrt{\gamma}}\left|\sqrt{12 - 3\gamma} - \sqrt{\gamma}\right| |u_0(x_0)| \qquad (1 \leq \gamma \leq 4). \tag{4.1}$$

This conclusion allow us to recover the result in [3]. A corollary in the same spirit as Corollary 4.1 holds true for $1 \leq \gamma \leq 4$. Outside this range it seems difficult to get purely "local-in-space" blowup criteria in the same spirit as in Theorem 2.1. But non local-in-space blowup conditions involving the computation of the $\|u_0\|_{H^1}$ as in (1.5) still apply outside the above range for the parameter γ.

As noted before, all these results apply in particular to periodic solutions: the statement of Theorem 2.1 remains valid for solutions $u \in C([0,T^*), H^s(\mathbb{S})) \cap C^1([0,T^*), H^{s-1}(\mathbb{S}))$, with $s > 3/2$, where $\mathbb{S}$ denotes the one-dimensional torus. It should be pointed, however, that the estimates of Lemma 3.2 (that are optimal for $u \in H^s(\mathbb{R})$, at least for a few specific choices of f and g) are no longer optimal when $u \in H^s(\mathbb{S})$. Improving such estimates would require the application of variational methods. Therefore, we expect that in the case of the torus, the restriction on the Lipschitz constant K appearing in Theorem 2.1 could be relaxed and the estimate on the coefficient β appearing in (2.1) improved. See [4] for results in this direction in the case of the classical period rod equation: the improvements on the convolution estimates specific to periodic solutions allow in particular to get rid of the unpleaseant restriction $1 \leq \gamma \leq 4$; the restriction to be put on γ in the case of the torus is that γ is outside a suitable neighborhood of the origin.

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